Mathware & Soft Computing 15 (2008) 95-111

Semigroups of Semi-copulas and Evolution of Dependence at Increase of Age

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Abstract

We consider a pair of exchangeable lifetimes X, Y and the families of the conditional survival functions $\overline{F}_t(x,y)$ of $(X-t, Y-t)$ given $(X>t, Y>t)$. We analyze some properties of dependence and of ageing for $\overline{F}_t(x,y)$ and some relations among them.

Keywords: copulas, semi-copulas, evolution of dependence under truncation, level curves of survival functions

1 Introduction

Let X, Y be exchangeable non-negative random variables and denote by $\overline{F}(x,y)$, $\overline{G}(x)$, $\hat{C}(u,v)$ the corresponding joint survival function, marginal univariate survival function and survival copula respectively; namely

$$\overline{F}(x,y) = P(X > x, Y > y), \tag{1}$$

$$\overline{G}(x) = \overline{F}(x,0) = P(X > x),$$

$$\hat{C}(u,v) = \overline{F}\left(\overline{G}^{-1}(u), \overline{G}^{-1}(v)\right). \tag{2}$$

We assume that $\overline{F}(x,y)$ is a continuous survival function which is strictly decreasing on $\mathbb{R}_+$ in each variable. This in particular implies that $\overline{G}(x)$ is a continuous, strictly decreasing survival function; we also assume $\overline{G}(0) = 1$.
Let us consider the function $B : [0,1] \times [0,1] \to [0,1]$ defined by

$$B(u,v) = \exp\left\{-\overline{G}^{-1}\left(\overline{F}(-\log u, -\log v)\right)\right\}. \tag{3}$$

It is immediate to see that B satisfies boundary conditions for a copula, is increasing in each variable and continuous, but it does not satisfy, generally, the rectangular inequality. For this reason we say that B is generally a semi-copula ([4, 10]). However, it turns out to be a copula in several cases of interest.

The function B can be used to describe certain "bivariate ageing" properties of the pair (X, Y) and has been called "bivariate aging function".

By imposing appropriate dependence conditions on B, it is possible to characterize some conditions of bivariate ageing for (X, Y); this can be used to analyze some relations existing among univariate ageing, bivariate ageing, and stochastic dependence (see [4]).

From a more technical point of view, relevant features of B are that it describes the family of the level curves of $\overline{F}$ and it permits to give a representation of $\overline{F}$ in terms of the pair $\left(\overline{G}, B\right)$, see Eq. (5) below.

An item of general interest is the conditional survival function:

$$\overline{F}_t\left(x, y\right) = P\left(X > t + x, Y > t + y | X > t, Y > t\right), \tag{4}$$

for $t > 0$. In fact the study of the evolution in time of the survival functions $\overline{F}_t\left(x, y\right)$ can be interesting in several fields (see e.g. [3, 5, 6, 15]).

As a natural consequence of the introduction of the family $\{\overline{F}_t\}_{t\geq 0}$, it is of interest to study the evolution of the families denoted by $\{B_t\}_{t\geq 0}$, $\{\hat{C}_t\}_{t\geq 0}$, with obvious use of the notation (see also Section 2).

In this paper we point out both analogies and structural differences between $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$.

Furthermore, in Section 2, we detail some relevant aspect of analytical type for $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$ respectively. We mention some practical interpretation of these analytical conditions, in particular about increase of dependence and ageing.

Section 3 is devoted to discuss specific results, along the lines indicated in [3, 4], related with evolution of dependence and bivariate ageing.

2 Some basic facts

First we briefly recall some of the arguments contained in [3, 4]. As immediate consequences of (2) and (3) respectively, we obtain

$$\overline{F}\ \left(x, y\right) = \hat{C}\{\overline{G}\left(x\right), \overline{G}\left(y\right)\},$$

$$\overline{F}\ \left(x, y\right) = \overline{G}\{-\log B\left(e^{-x}, e^{-y}\right)\}. \tag{5}$$

Furthermore we can easily obtain, still as a consequence of Eqs. (2) and (3), the following relations between B and $\hat{C}$:

$$B(u, v) = \exp\left[-\overline{G}^{-1}\{\hat{C}\left(\overline{G}(-\log u), \overline{G}\left(-\log v\right)\right)\}\right]; \tag{6}$$

$$\hat{C}(u, v) = \overline{G}\left\{-\log B\left(e^{-\overline{G}^{-1}(u)}, e^{-\overline{G}^{-1}(v)}\right)\right\}. \tag{7}$$

Remark 1. Notice that, if $\overline{G}(x) = e^{-x}$, i.e. if $\overline{G}(-\log u) = u$, then $B = \hat{C}$ and B is thus certainly a copula. More generally, we also observe that, if $\overline{G}(-\log u)$ is concave, then B is copula. This fact follows by the general method of transforming copulas by means of

$$C_\phi(u,v) = \phi^{-1}\big(C(\phi(u),\phi(v))\big),$$

with $\phi : [0,1] \to [0,1]$, ϕ bijective and concave (see e.g. [9, 12, 13, 14]).

Let us consider now the joint law of the residual lifetimes $(X-t, Y-t)$, conditional on the observation of the survival data $\{X > t, Y > t\}$. For $t > 0$ we put

$$\overline{F}_t\,(x,y) = P\,(X > t+x, Y > t+y | X > t, Y > t)\,,$$

$$\overline{G}_t\,(x) = P\,(X > t+x | X > t, Y > t)\,,$$

so that we can write

$$\overline{F}_t\,(x,y) = \frac{\overline{F}\ \ (x+t, y+t)}{\overline{F}\ \ (t,t)}, \tag{8}$$

$$\overline{G}_t\,(x) = \frac{\overline{F}\ \ (x+t, t)}{\overline{F}\ \ (t,t)}. \tag{9}$$

We are interested in studying the survival copula and the ageing function of $\overline{F}_t$. For this reason we need that $\overline{G}_t\,(x)$ is continuous and strictly decreasing on $\mathbb{R}_+$ in each variable. This is guaranteed by our assumption that $\overline{F}(x,y)$ is a continuous and strictly decreasing on $\mathbb{R}_+$. This assumption is also equivalent to $\hat{C}(u,v),\ B(u,v)$ being strictly increasing in u, for all $v \in (0,1]$.
For $0 \le u \le 1$, $0 \le v \le 1$, we then put

$$\hat{C}_t(u,v) \equiv \overline{F}_t\left(\overline{G}_t^{-1}\,(u)\,, \overline{G}_t^{-1}\,(v)\right) \tag{10}$$

and

$$B_t(u,v) \equiv \exp\left[-\overline{G}_t^{-1}\{\overline{F}_t\,(-\log u, -\log v)\}\right]. \tag{11}$$

Remark 2. For $t = 0$, B_t coincides with B as given in formula (3).

In view of (8) and (9), the relation between $\hat{C}_t$ and $\hat{C}$ has an explicit form given by

$$\hat{C}_t(u,v) = \frac{\hat{C}\left[\overline{G}\left(\overline{G}_t^{-1}(u)+t\right), \overline{G}\left(\overline{G}_t^{-1}(v)+t\right)\right]}{\hat{C}(\overline{G}(t), \overline{G}(t))}. \tag{12}$$

We notice that the function $\hat{C}_t$ in (12) is actually a copula for any $t > 0$.

Remark 3. $\hat{C}$ is strictly increasing in each variable if and only if B is strictly increasing in each variable.
$\hat{C}$ is strictly increasing in each variable if and only if $\hat{C}_t$ is strictly increasing in each variable u and v.
Hence B is strictly increasing in each variable if and only if B_t is strictly increasing in each variable u and v.

Remark 4. Notice that Eq. (12) contains the term $\overline{G}$. However, by adopting a different parametrization for $\hat{C}_t$, we realize that the family $\{\hat{C}_t\}_{t\geq 0}$ only depends on $\hat{C}$. In fact, by letting $\hat{C}_{(z)} := \hat{C}_{\overline{G}^{-1}(z)}$ we can write

$$\hat{C}_{(z)}(u,v) = \frac{\hat{C}\left(R(z,u\hat{C}(z,z)), R(z,v\hat{C}(z,z))\right)}{\hat{C}(z,z)}, \tag{13}$$

where

$$R(z,w) := \sup\{u|\hat{C}(u,z) \leq w\}$$

is the *residuum* of the copula $\hat{C}$ (see e.g. [8]).

The structure of the relation between B_t and B is radically different from the one binding $\hat{C}_t$ and $\hat{C}$. In fact it can only be given in an implicit form, as follows by Lemma 12 of [3]: $B_t\ (u,v)$ is such that

$$B\left(ue^{-t}, ve^{-t}\right) = B\left(B_t(u,v)e^{-t}, e^{-t}\right); \tag{14}$$

actually $B_t\ (u,v)$ is the unique solution σ of the equation

$$B\left(ue^{-t}, ve^{-t}\right) = B\left(\sigma e^{-t}, e^{-t}\right).$$

Eq. (14) will have in the following a basic role in proving some properties of $\{B_t\}_{t\geq 0}$.

Remark 5. For any $t > 0$, B_t only depends on B.

Other similarities between $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$ are shown by the following propositions, that will be used in Section 3.

Proposition 2.1. $\{B_t\}_{t\geq 0}$ *is a semigroup, i.e.*

$$(B_s)_r = (B_r)_s = B_{r+s} \qquad \forall t,s \geq 0.$$

Proof. In order to prove that $(B_r)_s = B_{r+s}$, in view of Eq. (14), we only have to check that

$$B_s(ue^{-r}, ve^{-r}) = B_s\left(B_{r+s}(u,v)e^{-r}, e^{-r}\right). \tag{15}$$

The latter is in fact the analog of the relation (14), with B replaced by B_s. On the other hand, by letting $t = r+s$ in Eq. (14), we can also write

$$B(ue^{-r-s}, ve^{-r-s}) = B\left(B_{r+s}(u,v)e^{-r-s}, e^{-r-s}\right). \tag{16}$$

Again using (14), for the left-hand side member of (16) we have

$$B(ue^{-r-s}, ve^{-r-s}) = B\left(B_s(ue^{-r}, ve^{-r})e^{-s}, e^{-s}\right); \tag{17}$$

similarly, for the right-hand side member of (16),

$$B\left(B_{r+s}(u,v)e^{-r-s}, e^{-r-s}\right) = B\left(B_s(B_{r+s}(u,v)e^{-r}, e^{-r})e^{-s}, e^{-s}\right). \tag{18}$$

Now, Eq. (14) shows that the left-hand side of (17) and the left-hand side of (18) are equal. Then the right-hand side of (17) and of (18) coincide. From the equality

$$B\left(B_s(ue^{-r}, ve^{-r})e^{-s}, e^{-s}\right) = B\left(B_s(B_{r+s}(u,v)e^{-r}, e^{-r})e^{-s}, e^{-s}\right),$$

Eq. (15) follows, since B is strictly increasing in each variable. □

The semigroup property also holds for the family $\{\hat{C}_t\}_{t\geq 0}$. We have in fact

Proposition 2.2. *$\{\hat{C}_t\}_{t\geq 0}$ is a semigroup.*

Proof. For fixed $r > 0$ we consider the survival model with joint survival function $\overline{F}_r$ and margin $\overline{G}_r$.
We now notice the semigroup property of the families $\{\overline{F}_t\}_{t\geq 0}$ and $\{\overline{G}_t\}_{t\geq 0}$. We can associate to the new model the families $\{(\overline{F}_r)_s\}_{s\geq 0}$ and $\{(\overline{G}_r)_s\}_{s\geq 0}$, such that, for any $t, s \geq 0$,

$$(\overline{F}_r)_s = \overline{F}_{r+s} \text{ and } (\overline{G}_r)_s = \overline{G}_{r+s}.$$

By definition, the survival copula of the model $\overline{F}_r$ is

$$\hat{C}^{(r)}(u,v) \equiv \overline{F}_r\left(\overline{G}_r^{-1}(u), \overline{G}_r^{-1}(v)\right) = \hat{C}_r(u,v).$$

We have to prove that

$$(\hat{C}^{(r)})_s = \hat{C}_{r+s} \qquad \forall r, s \geq 0.$$

By applying (12) to $(\hat{C}^{(r)})_s$, we obtain

$$\hat{C}_s^{(r)}(u,v) = \frac{\hat{C}^{(r)}\left[\overline{G}_r\left(\overline{G}_{r+s}^{-1}(u)+s\right), \overline{G}_r\left(\overline{G}_{r+s}^{-1}(v)+s\right)\right]}{\hat{C}^{(r)}(\overline{G}_r(s), \overline{G}_r(s))}. \tag{19}$$

By applying again (12) to $\hat{C}^{(r)}$ in Eq. (19),

$$\hat{C}_s^{(r)}(u,v) = \frac{\hat{C}\left[\overline{G}\left(\overline{G}_{r+s}^{-1}(u)+s+r\right), \overline{G}\left(\overline{G}_{r+s}^{-1}(v)+s+r\right)\right]}{\hat{C}(\overline{G}(r+s), \overline{G}(r+s))}. \tag{20}$$

The thesis follows by pointing out that the right-hand side of the last equation effectively coincides with $\hat{C}_{r+s}(u,v)$. □

Remark 6. We thought it is useful to present two independent proofs of Proposition 2.1 and Proposition 2.2. However, one could also obtain each proposition from the other one, by taking into account (6).
We notice that the proof of Proposition 2.2 does not require that $\hat{C}$ is strictly increasing in each variable, while the equivalent condition B strictly increasing in each variable is needed for the proof of Proposition 2.1.

As mentioned in the Introduction, one purpose of ours is to analyze increase or decrease of dependence between residual lifetimes. In this respect, we recall the following definitions.

Definition 2.3. Let $S_1,\ S_2$ be two semi-copulas. We write

$$S_1 \preceq S_2$$

iff

$$S_1(u,v) \leq S_2(u,v) \qquad \forall\ u,v \in [0,1].$$

Let (X_1,Y_1), (X_2,Y_2) be two random vectors and $\hat{C}_1,\ \hat{C}_2$ the survival copulas of (X_1,Y_1) and (X_2,Y_2) respectively.

Definition 2.4. (X_2,Y_2) is said *more concordant* than (X_1,Y_1) iff

$$\hat{C}_1 \preceq \hat{C}_2.$$

Considering $X_2 = X_1 - t$ and $Y_2 = Y_1 - t$, monotonicity of the mapping $t \mapsto \hat{C}_t$ has the following meaning: if $t \mapsto \hat{C}_t$ is increasing in t, the residual lifetimes will be more and more dependent as age increases.

We are then interested in describing analytical conditions for such monotonicity properties. In our parallel study of $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$, we are also interested in analytical conditions for monotonicity properties of the mapping $t \mapsto B_t$. To this purpose, we suppose $t \mapsto \hat{C}_t$ and $t \mapsto B_t$ differentiable (see below).

We denote, as usual, the common scalar product by $\cdot$, the gradient operator by ∇ and by $\dfrac{\partial}{\partial x_i}$ the partial derivative w.r.t. the i-th variable. Furthermore, since we are dealing with exchangeable variables and hence with symmetric survival functions and survival copulas, we can write

$$\nabla \hat{C}(z,z) \cdot (1,1) = 2\frac{\partial}{\partial x_1}\hat{C}(z,z) = 2\frac{\partial}{\partial x_2}\hat{C}(z,z).$$

Remark 7. The differentiability of $t \mapsto \hat{C}_t$ is guaranteed by the following conditions on $\hat{C}$:

- $\hat{C}(t,t) > 0$ for any $t \in (0,1]$,
- $\dfrac{\partial}{\partial x_1}\hat{C}(u,v),\ \dfrac{\partial}{\partial x_2}\hat{C}(u,v)$ exist and are strictly positive for any $(u,v) \in (0,1]^2$.

The differentiability of $t \mapsto B_t$ is guaranteed by the only existence and strictly positivity of $\dfrac{\partial}{\partial x_1}B(u,v),\ \dfrac{\partial}{\partial x_2}B(u,v)$ on the square $(0,1]^2$.

As we can expect in view of Remark 4, the monotonicity properties of $t \mapsto \hat{C}_t$ can be characterized in terms of $\hat{C}$ and its residuum. We have in fact

Proposition 2.5. *The mapping $t \mapsto \hat{C}_t$ is increasing iff*

$$2\left[\hat{C}(u,v) - \nabla\hat{C}(u,v)\cdot(u,v)\right] \geq \nabla\hat{C}(u,v)\cdot\left(\frac{\partial R}{\partial x_1}(1,u), \frac{\partial R}{\partial x_1}(1,v)\right), \qquad (21)$$

where R is the residuum of $\hat{C}$ and $\displaystyle\frac{\partial R}{\partial x_1}(1,u) \equiv \left.\frac{\partial R}{\partial x_1}(x_1,u)\right|_{x_1=1}$.

Proof. In view of the semigroup property of $\{\hat{C}_t\}_{t\geq 0}$, it is sufficient to study the sign of the derivative of $\hat{C}_t$ w.r.t. t for $t = 0$. Since the change of parameter given by $z = \overline{G}(t)$ (see Remark 4) is strictly decreasing, instead of differentiating Eq. (12) w.r.t. t, we can differentiate the simpler Eq. (13) w.r.t. z. Thus we need to check that

$$\left.\frac{\partial}{\partial z}\hat{C}_{(z)}(u,v)\right|_{z=1} \leq 0.$$

To this purpose, we now compute the partial derivative $\frac{\partial}{\partial z}\hat{C}_{(z)}(u,v)$.

$$\frac{\partial}{\partial z}\hat{C}_{(z)}(u,v) = \frac{1}{\hat{C}(z,z)^2}\Bigg\{\hat{C}(z,z)\nabla\hat{C}(R(z,u\hat{C}(z,z)), R(z,v\hat{C}(z,z)))\cdot$$

$$\cdot\left(\frac{dR}{dz}(z,u\hat{C}(z,z)), \frac{dR}{dz}(z,v\hat{C}(z,z))\right) - \hat{C}\left(R(z,u\hat{C}(z,z)), R(z,v\hat{C}(z,z))\right)\nabla\hat{C}(z,z)\cdot(1,1)\Bigg\},$$

where

$$\frac{dR}{dz}(z,u\hat{C}(z,z)) = \frac{\partial R}{\partial x_1}(z,u\hat{C}(z,z)) + u\frac{\partial R}{\partial x_2}(z,u\hat{C}(z,z))\left[\nabla\hat{C}(z,z)\cdot(1,1)\right].$$

Since $[\hat{C}(z,z)]^2$ is positive for any $z > 0$,

$$\frac{\partial}{\partial z}\hat{C}_{(z)}(u,v) \leq 0$$

iff

$$\hat{C}(z,z)\nabla\hat{C}(R(z,u\hat{C}(z,z)), R(z,v\hat{C}(z,z)))\cdot$$

$$\cdot\left(\frac{\partial R}{\partial x_1}(z,u\hat{C}(z,z)) + 2u\frac{\partial R}{\partial x_2}(z,u\hat{C}(z,z))\frac{\partial\hat{C}}{\partial x_1}(z,z), \frac{\partial R}{\partial x_1}(z,v\hat{C}(z,z)) + 2v\frac{\partial R}{\partial x_2}(z,v\hat{C}(z,z))\frac{\partial\hat{C}}{\partial x_1}(z,z)\right)$$

$$-2\hat{C}\left(R(z,u\hat{C}(z,z)), R(z,v\hat{C}(z,z))\right)\frac{\partial}{\partial x_1}\hat{C}(z,z) \leq 0. \qquad (22)$$

By putting $z = \overline{G}(0) = 1$ in Eq. (22), recalling that $\hat{C}(1,1) = 1$ and $\frac{\partial}{\partial x_1}\hat{C}(1,1) = 1$, by definition of copula, $R(1,w) = w$ and, consequently, $\frac{\partial R}{\partial x_2}(1,w) = 1$, we obtain

$$\nabla \hat{C}(u,v) \cdot \left(\frac{\partial R}{\partial x_1}(1,u) + 2u, \frac{\partial R}{\partial x_1}(1,v) + 2v \right) - 2\hat{C}(u,v)\frac{\partial \hat{C}}{\partial x_1}(1,1) \leq 0.$$

□

Remark 8. A sufficient condition for $z \mapsto \hat{C}_{(z)}$ being decreasing is

$$\nabla \hat{C}(u,v) \cdot \left(\frac{\partial R}{\partial x_1}(1,u) + 2u\frac{\partial R}{\partial x_1}(1,v) + 2v \right) \leq 0.$$

In fact, since $\hat{C}(z,z)$ is increasing in z, it is sufficient to impose the numerator of (13) decreasing in z.

Concerning the family $\{B_t\}_{t \geq 0}$, we have instead

Proposition 2.6. *$t \mapsto B_t$ is increasing if*

$$(u,v) \cdot \nabla B(u,v) \leq (B(u,v),1) \cdot \nabla B(B(u,v),1). \tag{23}$$

Proof. As in the previous proof, we have to compute the partial derivative of $B_t(u,v)$ w.r.t. t. Differentiating Eq. (14), we obtain

$$-ue^{-t}\frac{\partial}{\partial x_1}B(ue^{-t}, ve^{-t}) - ve^{-t}\frac{\partial}{\partial x_2}B(ue^{-t}, ve^{-t}) =$$

$$e^{-t}\left[\frac{\partial}{\partial t}B_t(u,v) - B_t(u,v)\right]\frac{\partial}{\partial x_1}B\left(B_t(u,v)e^{-t}, e^{-t}\right) - e^{-t}\frac{\partial}{\partial x_2}B\left(B_t(u,v)e^{-t}, e^{-t}\right).$$

Again, in view of the semigroup property of $\{B_t\}_{t \geq 0}$, we can restrict ourselves to study its sign only for a fixed t, e.g., for $t = 0$. We have

$$-(u,v) \cdot \nabla B(u,v) =$$

$$\frac{\partial}{\partial t}B_t(u,v)\frac{\partial}{\partial x_1}B\left(B(u,v),1\right) - B(u,v)\frac{\partial}{\partial x_1}B\left(B(u,v),1\right) - \frac{\partial}{\partial x_2}B\left(B(u,v),1\right).$$

Hence

$$\frac{\partial}{\partial t}B_t(u,v) = B(u,v) + \frac{\frac{\partial}{\partial x_2}B\left(B(u,v),1\right) - (u,v) \cdot \nabla B(u,v)}{\frac{\partial}{\partial x_1}B\left(B(u,v),1\right)}.$$

Since it is immediate by the definition of semi-copula that $\frac{\partial}{\partial x_1}B(u,v) \geq 0$ for any $u, v \in [0,1]$, $\frac{\partial}{\partial t}B_t(u,v) \geq 0$ when (23) holds. □

Concerning the condition $t \mapsto B_t$ increasing, we point out an aspect of the inequality $B_1 \preceq B_2$. This inequality can be equivalently expressed in terms of the level sets of the corresponding survival functions $\overline{F}_1$, $\overline{F}_2$.
For $z \in [0,1]$, let

$$L_z^{(\overline{F})} \equiv \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, \overline{F}(x,y) \geq z\}.$$

It can be easily shown that $B_1 \preceq B_2$ if and only if, for any $z \in [0,1]$,

$$L_{\overline{G}_1(z)}^{(\overline{F}_1)} \subseteq L_{\overline{G}_2(z)}^{(\overline{F}_2)}.$$

The semigroup property has interesting consequences in the analysis of $\{B_t\}_{t\geq 0}$ and $\{\hat{C}_t\}_{t\geq 0}$. The following Lemma will be applied to different situations in the next Section.

For a semi-copula S, let $\{S_t\}_{t\geq 0}$ be an arbitrary family of semi-copulas with the following properties:

- $S_0 = S$,
- $\{S_t\}$ is a semigroup, i.e., for any $r, s \geq 0$, $S_{r+s} = (S_r)_s$.

Lemma 2.7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two families of semi-copulas such that $\mathcal{C} \subseteq \mathcal{C}'$, i.e. $S \in \mathcal{C} \Rightarrow S \in \mathcal{C}'$.*
If $S \in \mathcal{C}$ is equivalent to $S_t \in \mathcal{C}'$ $\forall t \geq 0$, then $S \in \mathcal{C}$ implies $S_t \in \mathcal{C}$ $\forall t \geq 0$.

Proof. By hypothesis, $S \in \mathcal{C}$ implies $S_t \in \mathcal{C}'$ for all $t \geq 0$. In particular, $S_t \in \mathcal{C}'$ for all $t \geq r$ for any fixed $r \geq 0$. We can write $t = r + s$, for $s \geq 0$. By semigroup property, $S_t = (S_r)_s$ and, again by hypothesis, $(S_r)_s \in \mathcal{C}'$ for all $s \geq 0$ implies $S_r \in \mathcal{C}$. By the arbitrariness of r, the thesis follows.

□

This Lemma has substantially been used along the proof of Proposition 2.5 and can be similarly applied in proving Proposition 2.6. The same general fact can also turn out to be useful in the arguments of the next section.

3 Some properties of ageing and dependence and their relation

We start this Section by analyzing some relations between dependence and ageing properties along the same line of [4]. We recall that, for some families $\mathcal{C}$, the condition $B \in \mathcal{C}$ can be interpreted as a notion of bivariate ageing for $\overline{F}$ (see [3, 4]). To the purpose of a better understanding of both analogies and differences between the functions $\hat{C}$ and B, we introduce in the analysis here the notion of TP_2.

We recall that a function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be TP_2 (Totally Positive of order 2) if, for any $x' \le x'',\ y' \le y''$, it is

$$K(x'', y'')K(x', y') \ge K(x', y'')K(x'', y')$$

(see e.g [14] and references therein). Analogously a function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be RR_2 if, for any $x' \le x'',\ y' \le y''$, it is

$$K(x'', y'')K(x', y') \le K(x', y'')K(x'', y').$$

Moreover, a survival function $J : \mathbb{R}^+ \to (0, 1]$ is said to be IFR (Increasing Failure Rate) if J is log-concave and, conversely, DFR (Decreasing Failure Rate) if J is log-convex. J is NBU (New Better than Used) if, for any $x, y \ge 0$, $J(x+y) \ge J(x)J(y)$.

The following Proposition is analogous to some consequences of Propositions 5.2, 5.3, 5.4 of [4]. We however deal here with the concept of TP_2, that was not considered there; our proof is direct and independent of the results of [4].

Proposition 3.1.

1. *$\hat{C}$ TP_2, $\overline{G}$ IFR $\Rightarrow$ B TP_2.*
2. *B TP_2, $\hat{C}$ RR_2 $\Rightarrow$ $\overline{G}$ IFR.*
3. *B TP_2, $\overline{G}$ $DFR \Rightarrow$ $\hat{C}$ TP_2*

Proof. For simplicity sake let

$$x = -\log u,\ x' = -\log u',\ y = -\log v,\ y' = -\log v'$$

and

$$\alpha_{11} = \hat{C}(\overline{G}(x'), \overline{G}(y')),\quad \alpha_{12} = \hat{C}(\overline{G}(x'), \overline{G}(y)),$$
$$\alpha_{21} = \hat{C}(\overline{G}(x), \overline{G}(y')),\quad \alpha_{22} = \hat{C}(\overline{G}(x), \overline{G}(y)),$$

where $x' < x$ and $y' < y$. Thus we have

$$\alpha_{22} < \alpha_{12},\ \alpha_{21} < \alpha_{11}$$

and

$$-\log \alpha_{22} > -\log \alpha_{12},\ -\log \alpha_{21} > -\log \alpha_{11}.$$

1. In view of the adopted notation, the assumption $\hat{C}$ TP_2 becomes

$$\alpha_{11}\alpha_{22} \ge \alpha_{12}\alpha_{21}$$

or, equivalently,

$$\log \alpha_{11} - \log \alpha_{12} \ge \log \alpha_{21} - \log \alpha_{22}.$$

Furthermore, since $\overline{G}$ is IFR,

$$D^{-1}(x) = \overline{G}^{-1}(e^{-x})$$

is concave and increasing in x.
Thus, applying $D^{-1}(\cdot)$ to $-\log\alpha_{ij},\ i,j=1,2$, we obtain
$D^{-1}(-\log\alpha_{12})-D^{-1}(-\log\alpha_{11})\geq$

$$\geq D^{-1}(-\log\alpha_{22})-D^{-1}(-\log\alpha_{21})$$

and hence

$$\overline{G}^{-1}(\alpha_{11})+\overline{G}^{-1}(\alpha_{22})\leq\overline{G}^{-1}(\alpha_{12})+\overline{G}^{-1}(\alpha_{21}). \tag{24}$$

This is equivalent to $B\ TP_2$, in fact we can rewrite (24) as

$$-\overline{G}^{-1}(\alpha_{11})-\overline{G}^{-1}(\alpha_{22})\geq-\overline{G}^{-1}(\alpha_{12})-\overline{G}^{-1}(\alpha_{21}).$$

By applying the exponential to both the members, we obtain

$$e^{-\overline{G}^{-1}(\alpha_{11})}e^{-\overline{G}^{-1}(\alpha_{22})}\geq e^{-\overline{G}^{-1}(\alpha_{12})}e^{-\overline{G}^{-1}(\alpha_{21})},$$

that is

$$B(u,v)B(u',v')\geq B(u,v')B(u',v).$$

2. By the assumption $\hat{C}\ RR_2$

$$\alpha_{11}\alpha_{22}\leq\alpha_{12}\alpha_{21}.$$

Thus, by putting

$$u_{ij}:=-\log\alpha_{ij},\quad i,j=1,2,$$
$$u_{22}-u_{21}\geq u_{12}-u_{11}, \tag{25}$$

with $u_{22}>u_{21},\ u_{12}>u_{11}$. Furthermore, since the assumption B is TP_2, Eq. (24) holds, or, equivalently,

$$D^{-1}(u_{12})-D^{-1}(u_{11})\geq D^{-1}(u_{22})-D^{-1}(u_{21}). \tag{26}$$

By (25) and since $D^{-1}(x)$ is increasing in x, this last inequality holds only if $D^{-1}(x)$ is concave in x, that is $\overline{G}$ is IFR.

3. We have to prove now that

$$\alpha_{11}\alpha_{22}\geq\alpha_{12}\alpha_{21},$$

that is equivalent to

$$u_{22}-u_{21}\leq u_{12}-u_{11}.$$

Since $\overline{G}$ is DFR, $D^{-1}(x)$ is increasing and convex in x. Thus, if

$$u_{22}-u_{21}>u_{12}-u_{11},$$

we should have

$$D^{-1}(u_{12})-D^{-1}(u_{11})<D^{-1}(u_{22})-D^{-1}(u_{21}),$$

that contradicts Eq. (26) and therefore the hypothesis that B is TP_2.

$\square$

From now on we expand on ideas contained in [3]. For specific families $\mathcal{C}$ of semi-copulas, we analyze and compare conditions, on a survival model $\overline{F}$, of the type

$$\hat{C}_t \in \mathcal{C} \quad \forall t \geq 0,$$

$$B_t \in \mathcal{C} \quad \forall t \geq 0.$$

More precisely, the families that will be considered are the following:

the families of PQD and NQD exchangeable semi-copulas; we recall that a semi-copula S is PQD or NQD if it is $S\,(u,v) \geq u \cdot v$ or $S\,(u,v) \leq u \cdot v$, respectively;

the families of LTD and LTI exchangeable semi-copulas; we recall that a semi-copula S is LTD or LTI if $\dfrac{S\,(u,v)}{u}$ is non-increasing or non-decreasing in u, respectively;

the families of exchangeable semi-copulas $\mathcal{P}_{+}^{(3)}$ ($\mathcal{P}_{-}^{(3)}$), that were considered in [3] and [4], defined by the inequality

$$S(us,v) \geq S(u,vs) \qquad (S(us,v) \leq S(u,vs)), \qquad 0 \leq v \leq u \leq 1;$$

the families of TP_2 and RR_2 exchangeable semi-copulas, that were already mentioned.

Fix now a family $\mathcal{C}$ among those listed just above. We notice that, in view of Remark 4, $\overline{F}$ being such that $\hat{C}_t \in \mathcal{C}$ for all $t \geq 0$ is just a condition on $\hat{C}$; since we interpret $\hat{C} \in \mathcal{C}$ as a condition of dependence, we can also interpret $\{\hat{C}_t \in \mathcal{C}\ \forall t \geq 0\}$ as a condition of dependence; such a condition being typically stronger than $\hat{C} \in \mathcal{C}$.

Similarly, in the spirit of [4] and in view of Remark 5, we can interpret $\{B_t \in \mathcal{C},\ \forall t \geq 0\}$ as a condition of bivariate ageing.

In this way we can say that we are introducing here some potentially new notions of dependence and of bivariate ageing and want to analyze the relations existing among them.

As far as notions of dependence are concerned, this approach of defining potentially new conditions of dependence has been developed more systematically in the recent paper [7]. Here we rather develop and extend the approach in [3] by means of a few new remarks and results.

Let us start by considering the notion of PQD.

Proposition 3.2. *The condition $\hat{C}_t$ PQD for all $t \geq 0$, is equivalent to*

$$\hat{C}(u,u)\hat{C}(u'',v'') \geq \hat{C}(u,v'')\hat{C}(u,u''), \tag{27}$$

for any u, u'', v'' such that $0 \leq u'' \leq u \leq 1,\ 0 \leq v'' \leq u \leq 1$.

Proof. It is well known (and immediate to check) that the survival copula of a bivariate survival function $\overline{M}$ is PQD if and only if it is

$$\overline{M}(x,y) \geq \overline{G}_{\overline{M}}(x) \cdot \overline{G}_{\overline{M}}(y).$$

By taking $\overline{M} = \overline{F}_t$ we then see, in view of (4), that $\hat{C}_t$ PQD for all $t \geq 0$ means

$$\frac{\overline{F}\,(t+x, t+y)}{\overline{F}\,(t,t)} \geq \frac{\overline{F}\,(t+x, t)}{\overline{F}\,(t,t)} \frac{\overline{F}\,(t, t+y)}{\overline{F}\,(t,t)}$$

for any $t, x, y \geq 0$,

that can also be written in the form

$$\hat{C}\left(\overline{G}(t), \overline{G}(t)\right) \hat{C}\left(\overline{G}(t+x), \overline{G}(t+y)\right) \geq \hat{C}\left(\overline{G}(t+x), \overline{G}(t)\right) \hat{C}\left(\overline{G}(t), \overline{G}(t+y)\right).$$

By the arbitrariness in the choice of $t, x, y \geq 0$, the proof can be completed by letting

$$u = \overline{G}(t),\ u'' = \overline{G}(t+x),\ v'' = \ \overline{G}(t+y).$$

□

We notice that the condition in (27) is weaker than $\hat{C}$ TP_2 and strictly implies $\hat{C}$ PQD.

As to the family of LTD semi-copulas we can state

Proposition 3.3. *The condition $\hat{C}_t$ LTD for all $t \geq 0$, is equivalent to $\hat{C}$ being TP_2.*

Proof. It is also well known (and, again, immediate to check) that the survival copula of a bivariate survival function $\overline{M}$ is LTD if and only if it is

$$\frac{\overline{M}(x,y)}{\overline{G}_{\overline{M}}(x)} \text{ non-decreasing in } x, \text{for any } y \geq 0.$$

By taking again $\overline{M} = \overline{F}_t$ we then see, in view of (4), that $\hat{C}_t$ LTD for all $t \geq 0$, means

$$\frac{\overline{F}\,(t+x'', t+y)}{\overline{F}\,(t+x'', t)} \geq \frac{\overline{F}\,(t+x', t+y)}{\overline{F}\,(t+x', t)}$$

for any $t, x', x'', y \geq 0$, with $x'' > x'$.

By the arbitrariness of t, x', y and the condition $x'' > x'$, we can easily see that the above inequality is equivalent to the TP_2 property of $\overline{F}$. □

As to the condition $\hat{C}_t$ TP_2 for all $t \geq 0$, we have the following

Proposition 3.4. *$\hat{C}_t$ TP_2 for all $t \geq 0$ is equivalent to $\hat{C}$ TP_2.*

Proof. It is known (see e.g. [14]) that, for any fixed $t \geq 0$,

$$\overline{F}_t\ TP_2 \ \Leftrightarrow\ \hat{C}_t\ TP_2.$$

But, since

$$\overline{F}\ TP_2 \ \Rightarrow\ \overline{F}_t\ TP_2 \ \ \forall t \geq 0,$$

as straightly follows by the definitions of $\overline{F}_t$ and TP_2, it is sufficient $\hat{C}$ TP_2 to conclude that

$$\hat{C}_t\ TP_2 \ \ \forall t \geq 0.$$

□

Remark 9. An alternative proof of Proposition 3.4 can also be easily obtained by taking into account Proposition 3.3 and Lemma 2.7. In fact, $\hat{C}$ TP_2 implies $\hat{C}$ LTD and $\hat{C}$ is TP_2 if and only if $\hat{C}_t$ is LTD for all $t \geq 0$. Therefore, the hypotheses of the Lemma are verified for $\mathcal{C}$ and $\mathcal{C}'$ defined as the family of all TP_2 copulas and the family of all LTD copulas respectively.

In the following, we want show that, also as far as B is concerned, we can find a family of semi-copulas $\mathcal{C}$ such that $B \in \mathcal{C}$ is equivalent to $B_t \in \mathcal{C}$ for all $t \geq 0$. To this purpose we compare conditions of the type $B \in \mathcal{C}$ and $\{B_t \in \mathcal{C}\ \forall t \geq 0\}$. As already mentioned, for suitable families $\mathcal{C}$, the condition $B \in \mathcal{C}$ describes a property of bivariate ageing for $\overline{F}$. In particular, we recall that the conditions $B \in \mathcal{P}_+^{(3)}$ and B PQD can be seen as bivariate notions of IFR and NBU respectively (see e.g. [4]). In this respect, it is useful to point out the following facts:

Lemma 3.5. *(see [4]) The condition*

$$B \in \mathcal{P}_+^{(3)} \tag{28}$$

is equivalent to $\overline{F}_t$ being Schur-concave.

Lemma 3.6. *(see [3]) The condition (28) is equivalent to B_t being PQD for all $t \geq 0$.*

By applying Lemma 2.7 to the family $\{B_t\}$, we immediately obtain

Corollary 3.7. *The condition (28) is equivalent to*

$$B_t \in \mathcal{P}_+^{(3)}\ \forall\ t \geq 0.$$

Proof. It follows straightforward by Lemmas 2.7 and 3.6, with $\mathcal{C} \equiv \mathcal{P}_+^{(3)}$ and $\mathcal{C}'$ the family of PQD semi-copulas. □

In view of the afore-mentioned "bivariate ageing" interpretations of the conditions $B \in \mathcal{P}_+^{(3)}$ and B PQD, we can read Corollary 3.7 as follows: $\overline{F}_t$ bivariate NBU for all $t \geq 0$ is equivalent to $\overline{F}$ being bivariate IFR.

For a fixed univariate survival function $\overline{H}(x)$, consider now

$$\overline{H}_t(x) = P(X > x + t | X > t) = \frac{\overline{H}(x + t)}{\overline{H}(t)}.$$

The following chain of equivalences is very well known and easy to check:

$$\overline{H}\ IFR\ \Leftrightarrow\ \overline{H}_t\ IFR\ \Leftrightarrow\ \overline{H}_t\ NBU\ \ \forall t \geq 0.$$

Remark 10. Let us consider the family of the Archimedean semi-copulas $\{A_t\}$ associated to the survival functions $\overline{H}_t$'s,

$$A_t(u, v) = \overline{H}_t\left[\overline{H}_t^{-1}(u) + \overline{H}_t^{-1}(v)\right].$$

Following the arguments in [1] and [4], we can say that A_t describes (univariate) ageing properties of $\overline{H}_t$, in the sense that positive ageing properties of $\overline{H}_t$ correspond to negative dependence properties of A_t.

The equivalence

$$\overline{H}\ IFR \ \Leftrightarrow\ \overline{H}_t\ NBU\ \forall t \geq 0$$

can be written in the form

$$A \in \mathcal{P}_{-}^{(3)} \Leftrightarrow\ A_t\ NQD\ \forall t \geq 0. \tag{29}$$

We notice that, as a straight consequence of the semigroup property of $\{\overline{H}_t\}$, $\{A_t\}$ too is a semigroup and (29) can be given a proof analogous to the one of Corollary 3.7.

Concerning the TP_2 property for B, we can see that $B_t\ TP_2$ for all $t \geq 0$ is not implied by $B\ TP_2$.

On the other hand, the property $B_t\ TP_2$ for all $t \geq 0$ is actually also stronger than condition (28). In fact, as an immediate consequence of Lemma 3.6 and of the fact that $TP_2\ \Rightarrow\ PQD$, we have

Corollary 3.8. $B_t\ TP_2\ \ \forall t \geq 0 \Rightarrow\ B \in \mathcal{P}_{+}^{(3)}$.

As we have seen from Proposition 3.1, the link between ageing and dependence properties is not immediate, in the sense that we cannot derive ageing properties from dependence ones only, nor viceversa: we need a further condition on univariate ageing.

By combining Proposition 3.1 with Corollary 3.8 and Proposition 3.4, we obtain a link between $\mathcal{P}_{+}^{(3)}$ property of B and TP_2 property of $\hat{C}$.

Corollary 3.9. $\hat{C}\ TP_2$, $\overline{G}_t\ IFR\ \Rightarrow\ B \in \mathcal{P}_{+}^{(3)}$.

Remark 11. While $B_t\ PQD$ for all $t \geq 0$ implies $B \in \mathcal{P}_{+}^{(3)}$, we saw that $\hat{C}_t$ PQD for all $t \geq 0$ is not enough to get $\hat{C}\ TP_2$. However, we can still express $\hat{C}\ TP_2$ as a PQD-condition on models of residual lifetimes. Consider the family $\{\overline{F}_{a,b}\}_{a,b\geq 0}$ of joint survival functions,

$$\overline{F}_{a,b}(x,y) := P(X-a > x, Y-b > y | X > a, Y > b) = \frac{\overline{F}(x+a, y+b)}{\overline{F}(a,b)}$$

(so that, with this notation, $\overline{F}_t(x,y) = \overline{F}_{t,t}(x,y)$) and the corresponding families

$$\left\{\overline{G}_{a,b}^{(X)}\right\}_{a,b\geq 0},\quad \left\{\overline{G}_{a,b}^{(Y)}\right\}_{a,b\geq 0},\quad \left\{\hat{C}_{a,b}\right\}_{a,b\geq 0}.$$

We can easily check that the following equivalences holds

$$\hat{C}_{a,b}\ PQD\ \Leftrightarrow\ \overline{F}_{a,b}\ PQD \ \text{ and }\ \hat{C}_{a,b}\ TP_2\ \Leftrightarrow\ \overline{F}_{a,b}\ TP_2;$$

moreover,

$$\overline{F}_{a,b}\ PQD\ \forall\ a,\ b\ \Leftrightarrow\ \overline{F}_{a,b}\ TP_2\ \forall\ a,\ b\ \Leftrightarrow\ \overline{F}\ TP_2.$$

Analogous results to those above can be easily formulated for the negative dependence properties NQD, LTI, $S \in \mathcal{P}_{-}^{(3)}$, RR_2, corresponding to PQD, LTD, $S \in \mathcal{P}_{+}^{(3)}$, TP_2 respectively.
By combining the above arguments, some statements analogous to those in Proposition 15 of [3] could also be obtained.

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